On the gravitational field of static and stationary axial symmetric bodies with multi-polar structure

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Abstract

We give a physical interpretation to the multi-polar Erez-Rozen-Quevedo solution of the Einstein Equations in terms of bars. We find that each multi-pole correspond to the Newtonian potential of a bar with linear density proportional to a Legendre Polynomial. We use this fact to find an integral representation of the γ function. These integral representations are used in the context of the inverse scattering method to find solutions associated to one or more rotating bodies each one with their own multi-polar structure.
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1 INTRODUCTION

The adequate description of the gravitation field of an astrophysical object has been an important subject in both relativistic and Newtonian gravity since their origin. The particular case of the gravity associated to an axially symmetric body has played a central role in this theme. In Newtonian theory the gravitational potential of an axially symmetric body can be always represented by its usual expansion in terms of Legendre polynomials (zonal harmonics). In general relativity we have that the Einstein equations for an axially symmetric spacetime [1],

$$ds^2 = -e^{2\psi}dt^2 + e^{2\gamma-2\psi}(dr^2 + dz^2) + r^2e^{-2\psi}d\varphi, \tag{1}$$

reduces to

$$\psi_{,rr} + \psi_{,r}/r + \psi_{,zz} = 0, \tag{2}$$

$$\gamma[\psi] = \int r[(\psi_{,r}^2 - \psi_{,z}^2)dr + 2\psi_{,r}\psi_{,z}dz], \tag{3}$$

where the functions ψ and γ depend only on r and z.

The simplicity of the Weyl equation is rather deceiving since the potential ψ that obeys the usual Newtonian Laplace equation has a different meaning in general relativity. The mono-polar solution of the Einstein equations (Schwarzschild's solution) is represented in these coordinates by a bar of constant linear density of length $2m$, i.e., a 'potential' whose multi-polar expansions contains all the multi-polar moments beyond the dipole. Newtonian images are useful in general relativity but need to be used carefully [2]. Erez and Rosen [3] found that in spheroidal coordinates

$$x = (R_+ + R_-)/(2\sigma), \;\; y = (R_+ - R_-)/(2\sigma), \tag{4}$$

$$R_+ = \sqrt{r^2 + (z+\sigma)^2}, \;\; R_- = \sqrt{r^2 + (z-\sigma)^2}, \tag{5}$$

with $\sigma = m$ and $x \geq 1$, $-1 \leq y \leq 1$, the potential ψ outside an isolate axially symmetric body can be written as

$$\psi = -\sum_{k=0}^{\infty} q_k Q_k(x) P_k(y), \tag{6}$$

where P_k and Q_k are the Legendre polynomials and the Legendre functions of the second kind, respectively, and q_k are constants. The mono-polar terms correspond exactly to the potential representative of the Schwarzschild bar ($q_0 = 1$, and $q_k = 0$, for $k \neq 0$). The particular case $q_0 = 1, q_1 = 0, q_2 = q$ and $q_k = 0$ for $k > 2$ is the solution usually associated to Erez-Rosen and studied in text books [4].

The function γ for the general solution (6) is rather formidable and was first computed by Quevedo [5]. Also the physical meaning of the multi-polar moments was clarified in [5]. This is a rather important point that can be settled after the work of Gürsel [6] that shows the relation between the physically oriented definition of moments due to Thorne [7] and collaborators and the more geometrically motivated definition of Geroch [8] and Hansen [9] that are of simpler computation [10]. Also the moments for a charged stationary generalization of the Erez-Rosen-Quevedo solution were studied [11].

The aim of this paper is to give a physical interpretation to each term in the multi-polar expansion (6) in terms of Newtonian potentials associated to bars like the one of the above mentioned Schwarzschild bar. This interpretation is achieved in terms of potentials associated with bars of length $2m$ with densities proportional to Legendre polynomials. This integral representation of the function ψ can be used to find a simple integral representation of the non-trivial function γ. Furthermore, an integral representation of the function F used to generate stationary solutions in the context of the inverse scattering method (ISM) [12] [13] can be easily found. An example of solution generated by this method is a multi-polar deformed Kerr-NUT solution. We also discuss a new solution of the Einstein equations that represents two or more Kerr-NUT solutions, each one with their own multi-polar deformations, located along an axis of symmetry.

In Sec. II, combining two identities involving Legendre functions we find the physical interpretation of (6) in terms of the potential associated to bars. We also study an integral representation of the γ function. In Section. III,

we give a summary of the main formulas of the ISM and we discuss solitonic solutions whose seeds are a field of multi-poles ($q_0 = 0, q_k \neq 0, k > 0$). Particular cases are a "deformed" Kerr solutions and multiple "deformed" Kerr solutions . We conclude in Sec. IV with a few remarks.

2 BARS OF VARIABLE DENSITY AND THE γ FUNCTION

In this section we show the relation between the potential ψ defined in (6) and the Newtonian potential associated to bars. The first step is the Jeffery identity [14] that relates Legendre functions of second kind (Q_n) with Ferrers functions of the first (P_m^n) and second kind (Q_n^m), also known as associated Legendre functions,

$$\int_{\pi}^{\pi} Q_n(\frac{\bar{x}\cos t + \bar{y}\sin t + i\bar{z}}{c})\cos mt dt = 2\pi\frac{(n-m)!}{(n+m)!}Q_n^m(ir)P_n^m(\cos\vartheta)\cos(m\varphi) \tag{7}$$

with

$$(\bar{x}, \bar{y}, \bar{z}) = c[(\bar{r}^2+1)^{\frac{1}{2}}\sin\vartheta\cos\varphi, (\bar{r}^2+1)^{\frac{1}{2}}\sin\vartheta\sin\varphi, \bar{r}\cos\theta]. \tag{8}$$

By doing $m = 0$, $i\bar{r} = u$, $\cos\vartheta = v$ in (8) and using Neumann formula,

$$Q_n(\zeta) = \frac{1}{2}\int_{-1}^{1}\frac{P_n(t)}{\zeta - t}dt, \tag{9}$$

we find

$$2\pi P_n(v)Q_n(u) = \frac{1}{2}\int_{-1}^{1}P_n(t')dt'\int_{-\pi}^{\pi}\frac{dt}{A + B\cos t + C\sin t}, \tag{10}$$

with

$$\begin{aligned} A &= uv - t', \\ B &= (1-u^2)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}\cos\varphi, \\ C &= (1-u^2)^{\frac{1}{2}}(1-v^2)^{\frac{1}{2}}\sin\varphi. \end{aligned} \tag{11}$$

Hence

$$Q_n(u)P_n(v) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t')dt'}{\sqrt{(u^2-1)(1-v^2)+(uv-t')^2}}. \tag{12}$$

Note that this identity is valid for any complex variables u and v. Now identifying (u, v) with the spheroidal coordinates (x, y),, and using the fact that the inverse of (4) is:

$$r = \sigma\sqrt{(x^2-1)(1-y^2)}, \quad z = \sigma xy, \tag{13}$$

we find

$$Q_n(x)P_n(y) = \frac{1}{2}\int_{-\sigma}^{\sigma} \frac{P_n(z'/\sigma)dz'}{\sqrt{r^2+(z-z')^2}}. \tag{14}$$

Since the Newtonian potential of a bar of length 2σ with linear density λ located symmetrically along the z-axis is

$$\Phi_N = -\int_{-\sigma}^{\sigma} \frac{\lambda(z')dz'}{\sqrt{r^2+(z-z')^2}}. \tag{15}$$

The density associated to the potential $-Q_n(x)P_n(y)$ is

$$\lambda_n(z) = \frac{1}{2}P_n(z/\sigma), \tag{16}$$

and the Newtonian mass,

$$m_n = \frac{1}{2}\int_{-\sigma}^{\sigma} P_n(z/\sigma)dz. \tag{17}$$

Note that $m_n = 0$ for $n \neq 0$ and $m_0 = \sigma = m$. In other words the mono-polar term is the only one that carries mass, fact that is not surprising.

A direct computation using the identities of [2] shows that the function γ can be written as

$$\gamma = \gamma_0 + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} q_l q_k \int_{-\sigma}^{\sigma} dz' \int_{-\sigma}^{\sigma} dz'' P_k(z'/\sigma)P_l(z''/\sigma)\frac{r^2+(z-z')(z-z'')}{4(z'-z'')^2 R_{z'} R_{z''}}, \tag{18}$$

where γ_0 is a constant and $R_{z'} = \sqrt{r^2 + (z - z')^2}$, etc. This is a singular integral representation, the integration need to be taken as the Cauchy principal part, i.e., $\lim_{\eta\to 0, \epsilon\to 0} \int_{-\sigma+\eta}^{\sigma-\eta} \int_{-\sigma+\epsilon}^{\sigma-\epsilon}$, this allows us to extract the finite part of the integral, we are left with a constant of the type $\ln(0)$ that can be absorbed in the constant γ_0. In summary, the series (6) can be interpreted as the infinite sum of the potentials associated with bars of equal length, $2\sigma = 2m$, and linear densities $\lambda_n(z) = P_n(z/\sigma)/2$.

3 STATIONARY SOLUTIONS

The vacuum Einstein equations for the stationary axial symmetric metric

$$ds^2 = g_{ab}(r,z)dx^a dx^b + e^{\nu}(dr^2 + dz^2), \tag{19}$$

with $(x^a) = (x^0, x^1) = (t, \varphi)$, and $det(g_{ab}) = -r^2$ are equivalent to [13],

$$(rg_{,r}g^{-1})_{,r} + (rg_{,z}g^{-1})_{,z} = 0, \tag{20}$$

$$\nu = -\ln r - \frac{1}{4}\int r[tr(g_{,r}g^{-1}_{,r} - g_{,z}g^{-1}_{,z})dr + 2tr(g_{,r}g^{-1}_{,z})]dz, \tag{21}$$

where $g \equiv (g_{ab})$ and $g^{-1}_{,r} \equiv (g^{-1})_{,r}$, etc.

Equation (20) is an integrable system of equations that is closely related to the principal sigma model [15]. Techniques to actually find solutions of these equations are Bäcklund transformations and the inverse scattering method, also a third method constructed with elements of the previous two is the "vesture method", all these methods are closely related [15]. The general metric that represents the nonlinear superposition of several Kerr solutions with a Weyl solution can be found by using the inverse scattering method [13]. We find the N soliton solution

$$g_{ab} = r^{-N}\prod_{s=1}^{N}\mu_s(g^0_{ab} - \sum_{k,l=1}^{4}((\Gamma^{-1})_{kl}N^{(k)}_a N^l_b/(\mu_k\mu_l)) \tag{22}$$

$$g^0_{ab} = diag(-e^{2\psi}, r^2 e^{-2\psi}) \tag{23}$$

$$\Gamma_{kl} = \frac{-\mu_k\mu_l C_0^{(k)}C_0^{(l)}e^{2(\psi-F_k-F_l)} + C_1^{(k)}C_1^{(l)}e^{-2(\psi-F_k-F_l)}r^2}{\mu_k\mu_l(r^2+\mu_k\mu_l)} \tag{24}$$

$$N_0^{(k)} = -C_0^{(k)}e^{2(\psi-F_k)},\ N_1^{(k)} = C_1^{(k)}(r^2/\mu_k)e^{-2(\psi-F_k)} \tag{25}$$

$$e^{\nu} = e^{\gamma-\psi}(c_N)^2 r^{-N^2/2}(\prod_{k=1}^{N}\mu_k)^{(N+1)} \prod_{\substack{k,\,l\,=\,1\\ k\,>\,l}}^{N}(\mu_k-\mu_l)^{-2}det\Gamma_{ab} \tag{26}$$

$$\mu_k = W_k - z + [(W_k-z)^2+r^2]^{1/2}. \tag{27}$$

One of most interesting case is the superposition of one black hole with a field of multi-poles, i.e., a two soliton solution (N=2). In this case the constants $C_0^{(k)}$ and $C_1^{(k)}$ are related to the usual constants by [12]

$$\begin{aligned} C_1^1C_0^2 - C_0^1C_1^2 &= \sigma, \quad C_1^1C_0^2 + C_0^1C_1^2 = m\\ C_1^1C_1^2 - C_0^1C_0^2 &= -b, \quad C_1^1C_1^2 + C_0^1C_0^2 = a \end{aligned} \tag{28}$$

And

$$\sigma^2 = m^2 + b^2 - a^2. \tag{29}$$

Also, the constants W_k are

$$W_1 = z_1 + \sigma, \quad W_2 = z_1 - \sigma. \tag{30}$$

The symbol c_N that appears in e^{ν} is an arbitrary constant. m, a, b and z_1 represent the mass, the angular momentum per unit of mass, the NUT parameter, and the position along the z-axis of the black hole, respectively. The functions F_k are the solutions of the system of equations [13],

$$(r\partial_r - \lambda\partial_z + 2\lambda\partial_\lambda)F = r\psi_r,$$

$$(r\partial_z + \lambda\partial_r)F = r\psi_z, \tag{31}$$

$$F|_{\lambda=0} = \psi, \tag{32}$$

evaluates along the poles μ_k, i.e., $F_k = F|_{\lambda=\mu_k}$, see also [13] for an integral representation of F.

In summary, the N-soliton solution (22)-(26)(even N) represents the superposition of a Weyl solution characterized by (ψ, γ), with $N/2$ rotating

bars (Kerr-NUT solutions) located on $z = z_k$ with masses m_k, angular momenta per unit of mass a_k and NUT charges b_k. These constants are related by equations like (28) and (29). The case N odd was studied in [13].

The only integration that is left is the system of equations (31) with the initial condition (32). It is easy to check that the function

$$F = -\frac{1}{2}\sum_{k=1}^{\infty} q_k \int_{-\sigma}^{\sigma} \frac{P_k(z'/\sigma)(z' + R_{z'})}{(z' + R_{z'} + \lambda)R_{z'}} dz' \tag{33}$$

satisfies (31) and (32). We have excluded the mono-polar term due to the fact that the massive bar will appear as a result of the application of the ISM. The last expression can be obtained from the fact that the solution to (31)-(32) associated with the particular solution of Laplace equation $1/R_{z'}$ is $(z + R_{z'})/(z + R_{z'} + \lambda)$.

Now let us comeback to the two soliton solution. We have that the ISM produces a Kerr-NUT bar of length 2σ located along the z-axis with its center on $z = z_1$. Really the this bar is independent of the collection of bars that describes the multi-polar structure. To have a solution that represent an isolated body with a multi-polar structure we need to have bars of the same size located on the same position. This is achieved taking $z_1 = 0$ i.e., centering the Kerr bar on the origin of the z-axis and instead of taking $\sigma = m$ as in the static solution we need $\sigma^2 = m^2 + b^2 - a^2$. This solution is a special case of the one discussed in [11] that has electric charge.

The fact that the integral representation is written in cylindrical coordinates can be used to construct solutions representing two or more bodies each one with its independent multi-polar structure located along the z-axis in a simple way. In other words these coordinates allow us the separation of the different contributions without a considerable amount of work. In principle these configurations are not gravitationally stable, but the nonlinear interaction produces struts and/or strings that keep the bodies apart, see for instance [16].

To be more specific the metric functions associated to two rotating bodies with multi-polar structure can be obtained from applying the ISM to

the Weyl solution

$$-\psi=\sum_{k=1}^{\infty}[q_k^{(1)}\int_{-\sigma_1}^{\sigma_1}\frac{P_k(z'/\sigma_1)}{R_{z'}^{(1)}}dz'+q_k^{(2)}\int_{-\sigma_2}^{\sigma_2}\frac{P_k(z'/\sigma_2)}{R_{z'}^{(2)}}dz'], \tag{34}$$

where

$$R_{z'}^{(k)}=\sqrt{r^2+(z-z_k-z')^2}. \tag{35}$$

Thus (34) represents two collections of massless bars each one formed by bars of equal size $2\sigma_1$ and $2\sigma_2$ located along the z-axis around the points $z=z_1$ and $z=z_2$. The respective function F is,

$$-2F=\sum_{k=1}^{\infty}[q_k^{(1)}\int_{-\sigma_1}^{\sigma_1}\frac{P_k(z'/\sigma)(z'+R_{z'}^{(1)})}{(z'+R_{z'}^{(1)}+\lambda)R_{z'}^{(1)}}dz'+q_k^{(2)}\int_{-\sigma_2}^{\sigma_2}\frac{P_k(z'/\sigma)(z'+R_{z'}^{(2)})}{(z'+R_{z'}^{(2)}+\lambda)R_{z'}^{(2)}}dz'] \tag{36}$$

To obtain the two Kerr bars we construct a four soliton solution $N=4$, in this case we will have eight constants C_a^k that are related to the usual constants by [17]

$$C_1^1C_0^2-C_0^1C_1^2=\sigma_1,\quad C_1^1C_0^2+C_0^1C_1^2=m_1 \tag{37}$$

$$C_1^1C_1^2-C_0^1C_0^2=-b_1,\quad C_1^1C_1^2+C_0^1C_0^2=a_1 \tag{38}$$

$$C_1^3C_0^4-C_0^3C_1^4=\sigma_2,\quad C_1^3C_0^4+C_0^3C_1^4=m_2 \tag{39}$$

$$C_1^3C_1^4-C_0^3C_0^4=-b_2,\quad C_1^3C_1^4+C_0^3C_0^4=a_2. \tag{40}$$

And

$$\sigma_1^2=m_1^2+b_1^2-a_1^2,\quad \sigma_2^2=m_2^2+b_2^2-a_2^2 \tag{41}$$

Also, the constants W_k are

$$W_1=z_1+\sigma_1,\quad W_2=z_1-\sigma_1 \tag{42}$$

$$W_3=z_2+\sigma_2,\quad W_4=z_2-\sigma_2 \tag{43}$$

The generalization for the case of $n=N/2$ isolated bodies each one with their own multi-polar structure is obvious.

4 DISCUSSION

Multiple Kerr solutions with or without multi-polar structure has been discussed by several authors in different contexts and using a variety of solution generating techniques [18]. The four soliton representation of two Kerr solutions each one with their own multipolar deformations presented here has a clear meaning. Note that the multi-polar structure of the solution is generated by the multi-poles of the Erez-Rosen-Quevedo solution that have a precise meaning in general relativity.

For the single deformed Kerr solution [solution (22)-(27) with $N = 2$ and papameters (28)-(30)] as well as for the static case we did not study the elimination of conical singularities. The ISM like other similar methods introduce singular transformations like $t \to t + c\varphi$ via rotation of tetrads. This point together with the study of the horizons, the stability of the two body configuration and its related struts and rotating strings will be studied in a wider context. The study of struts and strings associated to two Kerr metrics can be found in [17]. Even though the strut are not physical objects (they have negative energy density) they can be used to extract useful information like the rate of emission of gravitational radiation of head-on collision of rotating black holes [19].

When we choose the constants C_a^k in such a way tha the NUT parameters b_k are zero the ISM, for even N, generates asymptotically flat solutions as long as the seed solutions are also asymptotically flat [13]. This is the case for the solutions generated from (6). Again in this analysis conical singularities are excluded.

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